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WITH FIAT MONEY

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**College of Commerce and Business Administration**  
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## I. INTRODUCTION

L. Shapley in collaboration with M. Shubik has presented several variants of a pure exchange economy model where money (fiat or commodity) serves as the medium of exchange. More specifically, each commodity is exchanged for money in a market where buyers bid money and sellers offer quantities of the commodity. Each buyer receives the portion of the aggregate amount of the commodity equal to the proportion of his bid to the aggregate bids, and vice versa for sellers.

This mechanism of exchange has several advantages over the Walrasian market mechanism. In a "thin" market, the mechanism does not presume price taking behavior among traders. On the other hand, the optimizing behavior of a trader in a market with many small traders is essentially that of a price taker. Hence, the difference between oligopolistic and competitive behavior is determined endogeneously. Another advantage of this model is that it distinguishes between feasible and optimal actions of economic agents. In the Walrasian analysis, there is no room for non-optimal behavior. Finally, the role of money and credit is naturally incorporated in the model. For more details, see papers by Shapley and





by Shubik in the references.

The purpose of this paper is to analyze the efficiency properties of equilibria resulting from the Shapley-Shubik mechanism in a perfectly competitive economy with fiat money, i.e., money without any intrinsic value. The purely competitive feature of the economy is captured by representing economic agents by an atomless measure space. Even then, Nash equilibria of such an economy are not always efficient. The main results of this paper consist of a characterization of those Nash equilibria which are Pareto efficient and their relations to the Walras equilibria in Aumann's market with a continuum of traders. Since most of the variants of Shapley's model [2] are meaningless with fiat money (as opposed to commodity money), a different variant is discussed in this paper. The description of an economy in this paper is that of Aumann's with the addition of an initial allocation of money agents. In this respect, we have a straightforward extension of the Shapley-Shubik model to a market with a continuum of traders. We interpret the initial allocation of money as a credit from bank which must be returned after trade ends. In this sense, the model is different from those in [2] where traders accept money for consumption. The usual interpretation of Nash equilibrium (which is relevant here) is that each player realizes his expectations about the behavior of others resulting from playing the same game repeatedly. In this interpretation, it is natural to assume that an agent ends up with the same amount of money he started with in order to play the same game again. This interpretation makes it plausible to restrict our attention to Nash equilibria which are Pareto efficient.

In Section II, the model is explained and our main results are stated. Proofs for these results as well as an example are given in Section III.



## II. THE EXCHANGE GAME AND MAIN RESULTS

An economic environment  $E$  is a quadruple  $\{T, R_+^n, (u_t)_{t \in T}, \hat{w}\}$

$T$  is the set of traders, assumed to be the unit interval endowed with the Lebesgue measure  $\lambda$ ;  $R_+^n$  is the commodity space and each trader's consumption possibility set;  $u_t : R_+^n \rightarrow R_+$  is the utility function of trader  $t$ ;  $\hat{w} : T \rightarrow R_+^n$  is a non-negative vector-valued function with  $\hat{w}(t)$  representing the initial endowment of trader  $t$ , and each coordinate of  $\hat{w}$ ,  $\hat{w}^i$  where  $\hat{w} = (\hat{w}^1, \dots, \hat{w}^n)$ , is integrable. We also introduce  $\hat{\mu} : T \rightarrow R_+$ , an integrable non-negative valued function representing the initial endowment of money to the traders.  $\mathcal{M}$  is the set of all initial profiles of money, i.e.,  $\mathcal{M} = \{\hat{\mu} : T \rightarrow R_+ | \hat{\mu} \text{ is integrable}\}$ .

Let  $M_i$  ( $i = 1, \dots, n$ ) be the market where good  $i$  is traded.

Trader  $t \in T$  chooses a strategy which is a signal sent to each market. A signal to  $M_i$  by  $t$ ,  $s_t^i = (x_t^i, z_t^i)$ , is a pair of non-negative real numbers with  $z_t^i$  being the amount of good  $i$  trader  $t$  intends to supply to  $M_i$  and  $x_t^i$  being the amount of money  $t$  pays for the amount of  $i$ -th good he wishes to purchase. Trader  $t$ 's individual signal is a pair of  $n$  vectors  $s_t = (x_t, z_t) = (x_t^1, \dots, x_t^n, z_t^1, \dots, z_t^n)$ .

A T-signal to  $M_i$ ,  $\hat{\sigma}^i = (\hat{\chi}^i, \hat{\xi}^i)$ , is a pair of real-valued integrable functions each of which maps  $T$  to  $R_+$ , representing the profile of signals sent to  $M_i$  over all traders.  $\hat{\sigma} = (\hat{\chi}, \hat{\xi}) = (\hat{\chi}^1, \dots, \hat{\chi}^n, \hat{\xi}^1, \dots, \hat{\xi}^n)$  is a T-signal and we denote  $\int \hat{\sigma} = (\int \hat{\chi}, \int \hat{\xi}) = (\int \hat{\chi}^1, \dots, \int \hat{\chi}^n, \int \hat{\xi}^1, \dots, \int \hat{\xi}^n)$  where  $\int \hat{\chi}^i = \int_T \hat{\chi}^i(t) d\lambda$  and  $\int \hat{\xi}^i = \int_T \hat{\xi}^i(t) d\lambda$ .

Given T-signals  $\hat{\sigma}$ , each market  $M_i$  allocates the  $i$ -th good and money it receives to traders according to their individual signals. While





the actual market allocation mechanism will be explained later, we denote the amount of  $i$ -th good trader  $t$  receives from  $M_i$  if  $T$ -strategy  $\hat{\sigma}$  prevails and  $t$  chooses his individual strategy  $s_t$  by  $d_t^i(\hat{\sigma}, s_t)$  and the amount of money he receives from  $M^i$  under the same condition by  $\ell_t^i(\hat{\sigma}, s_t)$ . Let  $d_t(\hat{\sigma}, s_t) = (d_t^1(\hat{\sigma}, s_t), \dots, d_t^n(\hat{\sigma}, s_t))$  and  $\ell_t(\hat{\sigma}, s_t) = \sum_{i=1}^n \ell_t^i(\hat{\sigma}, s_t)$ .

When choosing individual signals, trader  $t$  faces economic constraints. First, we assume he can send signals only once. Therefore, his signals must be bounded by his initial endowment  $\hat{w}(t)$  and his initial amount of money  $\hat{\mu}(t)$ . Namely, trader  $t$  can choose his individual signal  $s_t^i$  only from the following set

$$\alpha_t(\hat{\sigma}) = \left\{ s_t = (x_t, z_t) \in R_+^{2n} \left| \begin{array}{l} z_t \leq \hat{w}(t) \text{ and} \\ \sum_{i=1}^n x_t^i \leq \hat{\mu}(t) \end{array} \right. \right\}.$$

Second, money we wish to analyze in this model is fiat money, or money which possesses no intrinsic value. Hence, money is indispensable to make transactions (as will become clear later) but would become worthless once transactions took place. Therefore, we need a setup so that traders retain incentives to hold money after transactions are completed. For this purpose, we assume that each trader must pay prohibitive penalty if he ends up with a smaller amount of money than he is endowed with. Thus, nobody wants to choose signals which force him to pay such a penalty. Trader  $t$ , therefore, will choose his individual signals from

$$\beta_t(\hat{\sigma}) = \{s_t = (x_t, z_t) \in R_+^{2n} \mid \ell_t(\hat{\sigma}, s_t) > \sum_{i=1}^n x_t^i\}.$$





We call individual signal  $s_t$  feasible if it is in  $\alpha_t(\hat{\sigma}) \cap \beta_t(\hat{\sigma})$ .

One might justify the above formulation by the following story.

The initial endowment of money takes the form of an interest free loan from the central bank and must be repaid at the end of trading unless traders pay a prohibitive penalty. An alternative interpretation is that the economy we are analyzing is of a dynamic nature. The economy repeats period after period and we are only interested in a stationary state. We need to obtain the same amount of money holdings both at the beginning and at the end of the period for this purpose. This second interpretation enables us to discuss long-run equilibrium.

We consider market  $M^i$  of the following type. Provided that  $t$ 's individual signal is feasible,  $M^i$  allocates money and  $i$ -th good so that the proportion of money  $t$  receives to that traders altogether receive from  $M^i$  is equal to the ratio of  $t$ 's supply signal to the economy's aggregate supply signal to  $M^i$ . The proportion of  $i$ -th good  $t$  receives to all traders' receipt equals the ratio of  $t$ 's demand signal to the aggregate signal. That is

$$\ell_t^i(\hat{\sigma}, s_t) = \begin{cases} \frac{\int \hat{\chi}^i}{\int \hat{\xi}^i} z_t^i & \text{if } s_t \in \alpha_t(\hat{\sigma}) \cap \beta_t(\hat{\sigma}) \text{ and } \int \hat{\xi}^i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and



$$d_t^i(\hat{\sigma}, s_t) = \begin{cases} \frac{\int \hat{\xi}^i}{\int \hat{\chi}^i} x_t^i & \text{if } s_t \in \alpha_t(\hat{\sigma}) \cap \beta_t(\hat{\sigma}) \text{ and } \int \hat{\chi}^i = 0 \\ 0 & \text{otherwise} \end{cases}$$

In order to associate our model with the traditional Walras model, we might interpret  $\int \hat{\chi}^i / \int \hat{\xi}^i$  as the monetary price of the  $i$ -th good associated with  $T$ -strategy  $\hat{\sigma} = (\hat{\chi}, \hat{\xi})$ .

A consumption allocation  $\hat{\gamma} : T \rightarrow R^n$  is a vector valued integrable function such that  $\int \hat{\gamma} = \int \hat{w}$ . A consumption allocation  $\hat{\gamma}$  is called a consumption outcome under  $T$ -strategy  $\hat{\sigma}$  if

$$\hat{\gamma}(t) = \hat{w}(t) + d_t(\hat{\sigma}, \hat{\sigma}(t)) - \hat{\xi}(t)$$

a.e. in  $T$ .

Given this formulation, one can construct a game

$\Gamma = \{T, \Sigma(h_t)_{t \in T}\}$  where  $T$  is the set of all players which is the same  $T$  as before;  $\Sigma = R_+^{2n}$  is the strategy set common to all players.

Denote by  $\mathcal{S}$  the set of all  $T$ -strategies which are integrable, i.e.,

$\hat{\sigma} = (\hat{\chi}, \hat{\xi}) \in \mathcal{S}$  if  $\int \hat{\chi}^i < \infty$  and  $\int \hat{\xi}^i < \infty \forall i$ .  $h_t : \mathcal{S} \times \Sigma \rightarrow R_+$  is  $t$ -th player's payoff function such that

$$h_t(\hat{\sigma}, s_t) = u_t(\hat{w}(t) + d_t(\hat{\sigma}, s_t) - z_t)$$

for all  $t \in T$ ,  $\hat{\sigma} \in \mathcal{S}$  and  $s_t \in \Sigma$ .

Let us define the following.





Definition 1: A consumption allocation  $\hat{\gamma}$  and a vector  $p \in \mathbb{R}_+^n$  is a Walras equilibrium if a.e. in  $T$

$$p \cdot \hat{\gamma}(t) \leq p \cdot \hat{w}(t)$$

and  $u_t(\hat{\gamma}(t)) \geq u_t(c) \quad \forall c \text{ with } p \cdot c \leq p \cdot \hat{w}(t) \quad .$

Definition 2: A consumption allocation  $\hat{\gamma}$  is Pareto efficient if there is no other consumption allocation  $\hat{\gamma}'$  and a subset  $S \subseteq T$  with non-zero measure such that

$$u_t(\hat{\gamma}'(t)) \geq u_t(\hat{\gamma}(t)) \quad \text{a.e. in } T$$

and

$$u_t(\hat{\gamma}'(t)) > u_t(\hat{\gamma}(t)) \quad \forall t \in S \quad .$$

Definition 3: A T-strategy  $\hat{\sigma} \in \mathcal{X}$  is a Nash equilibrium if a.e.t in  $T$

$$\hat{\sigma}(t) \in \alpha_t(\hat{\sigma}) \cap \beta_t(\hat{\sigma})$$

and

$$h_t(\hat{\sigma}, \hat{\sigma}(t)) \geq h_t(\hat{\sigma}, s_t) \quad \forall s_t \in \Sigma_t \quad .$$

Under an assumption (A)  $u_t(\cdot)$  is continuous, concave and exhibits monotonicity for all  $t$ , we prove the following.

Theorem 1: For any Walras equilibrium for an economic environment  $E$ , there is a money assignment  $\hat{\mu} \in \mathcal{M}$  and a corresponding





Nash equilibrium,  $\hat{\sigma}$ , for  $(E, \hat{\mu})$  where the consumption outcome under  $\hat{\sigma}$ ,  $\hat{\gamma}$ , is the Walras allocation.

Corollary 1: For any economic environment  $E$ , there always exist a money endowment  $\hat{\mu}$  and a corresponding Nash equilibrium whose consumption outcome  $\hat{\gamma}$  is Pareto efficient.

Before stating another corollary, let us define the following.

Definition 4: A money endowment  $\hat{\mu}$  is called essentially positive if there exists  $\epsilon > 0$  such that  $\hat{\mu}(t) \geq \epsilon$  a.e.t in  $T$ .

Definition 5: A commodity endowment  $\hat{w}$  is called essentially bounded if for each  $i = 1, \dots, n$  there exists  $M_i > 0$  such that  $\hat{w}^i(t) \leq M_i$  a.e.t in  $T$ .

Corollary 2: For any economic environment  $E$  and an essentially positive money endowment  $\hat{\mu}$  and an essentially bounded  $\hat{w}$ , there always exist a Nash equilibrium  $\hat{\sigma}$  whose consumption outcome  $\hat{\gamma}$  is a Walras allocation for  $E$  and is Pareto efficient.

Therefore, if the initial money assignment is correctly chosen, our market mechanism can attain a Pareto efficient allocation. However, as will be shown in an example later, even if the correct profile of money (i.e., uniformly positive money endowment) is assigned, an inefficient Nash equilibrium may result. This interests one in characterizing the differences between efficient and inefficient Nash equilibria.

Definition 5: A Nash equilibrium T-strategy  $\hat{\sigma}$  is called  $\mathcal{M}$ -stable with respect to  $(E, \hat{\mu})$  if for almost all  $t \in T$ ,  $t$ 's best strategy  $s_t = \hat{\sigma}(t)$  does not change even if his initial money assignment  $\hat{\mu}(t)$  is increased.

Definition 6: A Nash equilibrium  $\hat{\sigma} = \dots(\hat{\chi}, \hat{\xi})$  is called non-trivial if  $\int \hat{\chi}^i \neq 0 \quad \forall i$  and  $\int \hat{\xi}^i \neq 0 \quad \forall i$ .



Theorem 2: The consumption allocation of a  $\mathcal{M}$ -stable non-trivial Nash equilibrium for  $(E, \hat{\mu})$  is a Walras equilibrium allocation for  $E$ .

Corollary 3: The consumption allocation of a  $\mathcal{M}$ -stable non-trivial Nash equilibrium is Pareto efficient.





## III. PROOFS

Proof of Theorem 1:

Let  $E$  be given. Let  $\hat{y}$  and  $p$  be a Walras equilibrium for  $E$ . Choose  $\hat{\mu} \in \mathcal{M}$  such that for a.e.  $t$  in  $T$ ,

$$\epsilon \cdot \sum_{i=1}^n p^i \cdot \text{Max} [\hat{y}^i(t) - \hat{w}^i(t), 0] \leq \hat{\mu}(t)$$

for some  $\epsilon > 0$ . Let

$$\hat{\chi}^i(t) = \epsilon \cdot p^i \cdot \text{Max} [\hat{y}^i(t) - \hat{w}^i(t), 0]$$

and

$$\hat{\xi}^i(t) = \text{Max} [0, \hat{w}^i(t) - \hat{y}^i(t)] .$$

Then obviously  $\hat{\sigma}(t) = (\hat{\chi}(t), \hat{\xi}(t)) \in \Sigma_t$  and  $\hat{\sigma}(t) \in \alpha_t(\hat{\sigma}) \cap \beta_t(\hat{\sigma})$ .

The market allocation rules give us the consumption outcome  $\hat{y}$  which is the Walras equilibrium. By Definition 1, it is a Nash equilibrium. Q.E.D.

Proof of Corollary 1:

Straightforward from Theorem 1 and assumption A. Q.E.D.

Proof of Corollary 2:

Straightforward from the proof of Theorem 1. Q.E.D.

Let us present an example which shows that the converse of Corollary 2 does not hold. Namely, let us show that even if  $\hat{\mu}$  is strictly positive, a Nash equilibrium may not correspond to a Walras equilibrium



Example: Let  $E$  be such that  $n = 2$ ,  $u_t(c^1, c^2) = c^1 \cdot c^2$  for all  $t \in T$ ,

$$\hat{w}(t) = \begin{cases} (0, 1) & \text{if } t \in [0, 1/2] \\ (1, 0) & \text{if } t \in (1/2, 1] \end{cases}.$$

Let  $\hat{\mu}(t) = 1/2$  for all  $t$ . Then  $E$  satisfies (A) and  $\hat{\mu}$  is strictly positive. It is easy to show that there is a Walras equilibrium  $(\hat{\gamma}, p)$  such that  $\hat{\gamma}(t) = (\frac{1}{2}, \frac{1}{2})$  for all  $t \in T$  and  $p = (p^1, p^2)$  with  $p^1 = p^2$ .

Equilibrium consumption allocation  $\hat{\gamma}$  is unique and equilibrium price vector  $p$  is unique up to a scalar multiplication. By Corollary 2, there exists a corresponding Nash equilibrium whose consumption outcome is  $\hat{\gamma}$ . Indeed, a T-strategy  $\hat{\sigma} = (\hat{\chi}, \hat{\xi})$  is a Nash equilibrium if

$$\hat{\chi}(t) = \begin{cases} (\frac{1}{2}, 0) & \text{if } t \in [0, \frac{1}{2}] \\ (0, \frac{1}{2}) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

and

$$\hat{\xi}(t) = \begin{cases} (0, \frac{1}{2}) & \text{if } t \in [0, \frac{1}{2}] \\ (\frac{1}{2}, 0) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$$

However, this is not the unique Nash equilibrium. For example, define  $\hat{\sigma}' = (\hat{\chi}', \hat{\xi}')$  such that for some  $\varepsilon$  with  $0 < \varepsilon < \frac{1}{2}$ ,





$$\hat{\chi}'(t) = \begin{cases} (\frac{1}{2}, 0) & \text{if } t \in [0, \frac{1}{2}] \\ (0, \frac{1}{2}) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

$$\hat{\xi}'(t) = \begin{cases} (0, \epsilon) & \text{if } t \in [0, \frac{1}{2}] \\ (\epsilon, 0) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

is a Nash equilibrium. The corresponding consumption outcome,  $\hat{\gamma}'$ , is

$$\hat{\gamma}'(t) = \begin{cases} (\epsilon, 1-\epsilon) & \text{if } t \in [0, \frac{1}{2}] \\ (1-\epsilon, \epsilon) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}.$$

This situation could be understood in the following way. Let us consider the usual Walras economy. The price  $p' = (\frac{2}{\epsilon}, \frac{2}{\epsilon})$  is the Walras equilibrium price. However, traders cannot transmit the signal corresponding to the Walras demand (or national demand) because they do not have enough amount of money.

Hence, the actual demand signal,  $\hat{\chi}'(t)$  (or effective demand) falls short of the Walras demand and hence the consumption outcome becomes non-Walrasian. This situation may be ameliorated if we allow traders to borrow money. If we do, traders will choose Walras signals instead and Walras consumption will result as a Nash equilibrium. In this sense, a trader's inclination to borrow is an important characteristics of non-Walrasian (and hence inefficient) Nash equilibrium.  $\mathcal{M}$ -stability defined in the previous section implies the non-existence of inclination to borrow.



Proof of Theorem 2:

Let  $\hat{\sigma} = (\hat{\chi}, \hat{\xi})$  be an  $\mathcal{M}$ -stable Nash equilibrium for  $(E, \hat{\mu})$  and  $\hat{\gamma}$  its consumption outcome. Let  $p \in R_+^n$  be such that  $p^i = \int \hat{\chi}^i / \int \hat{\xi}^i$  for all  $i$ .

Note that  $0 < p^i < \infty \quad \forall i$  because  $\hat{\sigma}$  is a non-trivial Nash Equilibrium. By the definitions of  $\alpha_t$ ,  $\beta_t$  and Nash Equilibrium, a.e.t in  $T$   $u_t(\hat{\gamma}(t)) \geq u_t(c)$  for all  $c$  such that (1)  $c \in R_+^n$ , (2)  $pc \leq p\hat{w}(t)$ , and (3)  $\sum_{i=1}^n p^i \cdot \text{Max} [0, c^i - \hat{w}^i(t)] \leq \hat{\mu}(t)$ .

However, since  $\hat{\sigma}$  is  $\mathcal{M}$ -stable, (3) is not an effective constraint for a.e.t in  $T$ . In other words,  $(p, \hat{\gamma})$  is a Walrasian equilibrium.  
Q.E.D.

Proof of the Corollary:

Straightforward from Theorem 2.





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